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LETTER TO THE EDITOR

Uniqueness on information theory and nuclear reactions

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Abstract. We show that the ensemble of scattering matrices defined by the Poisson kernel in the statistical theory of nuclear reactions, is uniquely determined under the analyticity-ergodicity requirement and maximisation of Shannon's entropy.

The maximal entropy (ME) approach has been extensively applied in the formulation of various theories (Jaynes 1957a, b, Wehrl 1978 and references therein). Basically, the idea behind this procedure is that the probability density function (PDF) determined with the entropy as the starting concept, corresponds to the least biased description of the problem on the basis of the available information. According to this procedure, a maximal entropy PDF consistent with the mean values of the n linearly independent but not necessarily commutating observables $A_1, A_2, \ldots, A_n$, is given by

$$p(\zeta)=\exp\left|-\sum_r \lambda_r A_r(\zeta)\right| \tag{1}$$

where the Lagrange multipliers must be fixed to assure the constraints.

On the other hand, we know that a PDF consistent with a given set of constraints could be determined by other methods different to the ME procedure. Assume that this is the case and a set of probability density functions, satisfying the same conditions, have been determined. In such a case, besides the goodness of the description, emerges the question related to the uniqueness of the ensemble defined by these PDFs. In this sense and in order to define the ensemble uniquely, the concept of maximal entropy would be of great importance. This is the kind of problem that we will discuss here in connection with the ensembles of scattering matrices that can be constructed in the statistical theory of nuclear reactions. More precisely, we present a theorem that establishes rigourously the uniqueness of the PDF that maximises Shannon's entropy and we use it to show that the successful ensemble of scattering matrices, analytically determined by Mello *et al* (1985), is uniquely defined.

Let us now consider the following theorem.

Theorem 1. The probability density function which maximises Shannon's entropy

$$S(p)=-\int_{\mathfrak{C}} p \ln p \, \mathrm{d}\mu \tag{2}$$

on the subset E_l of all the PDFs consistent with some linear constraints, is unique.

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Proof. We shall work with the functional

$$N(p)=\int_{\mathfrak{C}} p(\zeta)\ln p(\zeta)\,\mathrm{d}\mu(\zeta) \tag{3}$$

and prove that it has a unique minimum. N is defined for positive real valued functions $p(\zeta)$, ζ is a point in a complex domain denoted by $\mathfrak{C}$ and $\mathrm{d}\mu(\zeta)$ defines the measure.

It is well known, that the functional $N(p)$ is convex, then, for every two PDF r, s and $0\leqslant\lambda\leqslant 1$, it satisfies the relation

$$N(\lambda r+(1-\lambda)s)\leqslant\lambda N(r)+(1-\lambda)N(s). \tag{4}$$

Let us now assume that there are two PDF p and q on E_l which minimise the functional N globally. This means that

$$N(p)=N(q). \tag{5}$$

Starting from this equality we want to prove that p must be equal to q. It is convenient to define, for $0\leqslant\lambda\leqslant 1$, the function

$$g(\lambda)=N(\lambda p+(1-\lambda)q). \tag{6}$$

Due to the convexity of N, every PDF belonging to the interval $L(p,q)$ minimises N too. Thus, it is straightforward to show that the function g is constant. Therefore

$$g'(\lambda)=g''(\lambda)=0, \tag{7}$$

with

$$g'(\lambda)=\int_{\mathfrak{C}}(q-p)\ln[p+\lambda(q-p)]\,\mathrm{d}\mu(\zeta) \tag{8}$$

and

$$g''(\lambda)=\int\frac{(q-p)^2}{p+\lambda(q-p)}\,\mathrm{d}\mu(\zeta). \tag{9}$$

Since $[p+\lambda(q-p]$ is positive, we conclude that

$$(q-p)^2=0, \tag{10}$$

which proves the theorem.

As mentioned above, we are interested in the particular problem of the uniqueness of the ensemble of scattering matrices. We will discuss this point in connection with the ensemble defined by Mello *et al* in terms of the Poisson kernel

$$p(Z^*,\xi)=\frac{[\det(I-Z^*Z)]^{(n+1)/2}}{V|\det(I-Z^*\xi)|^{n+1}}, \tag{11}$$

which, among other PDFs, satisfies the so-called analyticity-ergodicity requirement or reproducing property expressed as

$$f(Z)=\int_{\mathfrak{C}} p(Z^*,\xi)f(\xi)\,\mathrm{d}\mu(\xi). \tag{12}$$

In these equations, $f(\xi)$ is any analytic function, Z is a point in the domain $\mathscr{R}$ of the complex and symmetric matrices (the optical S matrices), $\mathfrak{C}$ is the domain of the symmetric and unitary matrices ξ (the scattering matrices) and $d\mu(\xi)$ is the invariant measure under unitary transformations (Hua 1963).

According to the previous theorem, the ensemble of S matrices will be uniquely defined by the Poisson kernel if this PDF maximises Shannon's entropy. Prior to showing this, we make the following two remarks.

Remark 1. The Poisson kernel can be written as

$$p(Z^*, \xi) \propto |\exp[\Phi(Z^*, \xi)]|^2 = \exp[\phi(Z, \xi)], \tag{13}$$

with

$$\Phi(Z^*, \xi) = \frac{n+1}{2} \sum_r \frac{1}{k} \operatorname{Tr}(Z, \xi)^k, \tag{14}$$

an analytic function and

$$\phi(Z^*, \xi) = 2 \operatorname{Re} \Phi(Z^*, \xi). \tag{15}$$

Remark 2. For any two PDF, p and q, we have the relation (Tolman 1938)

$$p + q \ln q \geqslant q \ln p + q, \tag{16}$$

or equivalently

$$S(q) = -\int q \ln q \, d\mu \leqslant -\int q \ln p \, d\mu. \tag{17}$$

Using the preceding relations and the theorem 1, we can show the following theorem.

Theorem 2. Let p and q be two reproducing kernels in the sense of equation (12), and suppose that p can be written as

$$p(Z^*, \zeta) = \exp[\psi(Z^*, \zeta)] \tag{18}$$

with $\psi(Z^*, \zeta)$ an harmonic function, then, p maximises Shannon's entropy.

Proof. By hypothesis we have that

$$\int q \ln p \, d\mu = \int q\psi \, d\mu. \tag{19}$$

The reproducing property of q and p allows us to write the last equation as

$$\int q \ln p \, d\mu = \psi(Z^*, Z) = \int p\psi \, d\mu, \tag{20}$$

and so

$$\int q \ln p \, d\mu = \int p \ln p \, d\mu. \tag{21}$$

Taking into account the relation (17), we have, finally, that

$$S(q) = -\int q \ln q \, d\mu \leqslant -\int p \ln p \, d\mu = S(p), \tag{22}$$

which proves that the entropy associated to the PDF p is maximal.

In this way we have shown that the ensemble of scattering matrices is uniquely defined under the analyticity-ergodicity and maximal entropy requirements. In other words, the Poisson kernel defines uniquely the ensemble of scattering matrices.

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